Investigating Methods of Optimising the Shape of a Floating Lantern

IB Mathematics Higher Level - Math Exploration

Introduction

One of the most enchanting experiences I had was making a wish as my family set alight a floating lantern on New Year's Eve. Hence, their beauty has left a strong imprint on my mind. However, I was equally captivated by the prospect of exploring the mathematics behind their shape due to my long-standing interest in the subject. They are typically shaped like a lightbulb, so that hot gases can be held close to the top of the lantern to produce more lift [1]. This is shown in figure 1 below.

Figure 1 The typical shape of a floating lantern [2]

A similar shape is also seen in other objects such as helium balloons, which suggests that having a smaller radius at the base of a balloon is beneficial towards floating.

However, greater volume would produce more lift [3], which could allow the lantern to float longer and higher. At the same time, minimising surface area would reduce the weight and cost of the lantern. Hence, obtaining the ideal shape may be viewed as an optimisation between volume and surface area. Therefore, the aim of this investigation is to explore different mathematical models, which lead to different methods of optimising the surface area of a floating lantern.

To begin, it was necessary to set a scale since the principles discussed are also embodied by much larger objects such as hot air balloons. Hence, the mass of the floating lantern being optimised in this investigation was fixed based on lanterns I purchased, which weighed 0.034 kg each when deflated. Using this, I wanted to ascertain the minimum volume required for the lanterns to float. This is because the minimum volume needed would allow the surface area to be minimised the most while keeping the lanterns operational.

Figure 2 The floating lantern purchased for reference

To find the minimum volume, the following formula was used:

$$\rho_{object} < \rho_{air}$$

where 'ρ' represents density [4]. Here, it was assumed that the lantern would use paraffin wax, similar to the one that was purchased. Since paraffin wax releases CO_2, and burns at a temperature of about 200°C [5], the following assumptions were made:

- Density of air outside the lantern is 1.204 kg m^{-3} (at 1 atm pressure and 20°C) [6].
- Density of CO^2 inside the lantern is 1.118 kg m^{-3} (at 1 atm pressure and 200°C) [7].

[1] www.quora.com/Why-is-a-hot-air-balloon-shaped-like-a-light-bulb
[2] etvnews.com/pirates-den-hosts-lights-on-floating-lantern-event/
[3] www.engineeringtoolbox.com/hot-air-balloon-lifting-force-d_562.html
[4] physics.info/buoyancy/summary.shtml
[5] shamrockaffiliations.ws/Paraffin_Wax_Flash_Points.php#.Xke15y17FQI
[6] www.engineersedge.com/calculators/air-density.htm
[7] www.engineeringtoolbox.com/carbon-dioxide-density-specific-weight-temperature-pressure-d_2018.html?vA=200&degree=C&pressure=1bar#

Using this information, the calculation was continued. Here, 'm' is used to denote mass and 'V' is used to denote volume, which is to be found.

$$\rho_{object} < \rho_{air}$$

$$\frac{m}{V} < 1.204$$

The mass of the inflated lantern is the sum of the mass of the deflated lantern and the carbon dioxide inside the lantern. Therefore,

$$\frac{m_{lantern} + m_{CO2}}{V} < 1.204 \qquad [m_{CO2} = \rho_{CO2} \times V]$$

$$0.034 + 1.118V < 1.204V$$

$$V > \frac{0.034}{1.204 - 1.118}, \quad \boxed{V > 0.3953... \text{ m}^3}$$

As a result, the volume of the lantern was kept constant at 0.4 m³ for the rest of the investigation. After this, the aim was to construct mathematical models of floating lanterns to explore which could lead to an accurate optimisation.

To do this, each model will be adjusted based on two variables:

- A radius parameter for the lantern (r)
- A height parameter (h)

The way in which these parameters are defined and measured may be different for each model.

Model 1: Solid of revolution using a quadratic equation

Figure 3 Tracing over a lantern to find the solid of revolution

The most intuitive model that I could think of was a solid of revolution. This is because floating lanterns generally have cross-sections that are approximately circular. A solid of revolution is also likely to provide a simplistic method of optimisation.

To determine which function would accurately model a lantern, I traced over the image shown previously, and tried to recognise the function that could be rotated. This is shown in figure 3.

The edges of the lantern appear to align with a quadratic function, which is indicated by the black lines. The domain of this function is shown using purple lines.

The white line represents the position of largest radius (r) of the lantern. Visually, it may be inferred that approximately one fourth of the height (h) should be above the largest radius.

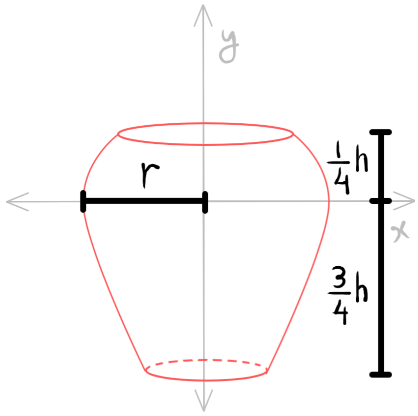

Figure 4 Defining parameters of the solid of revolution

(all diagrams drawn on Sketchpad 5.1[11])

Using this information, the model was planned as shown in figure 4, and modelled using the following function:

$$x = ry^2 - r; -0.75h < y < 0.25h$$

'r' was added as coefficient to 'y^2' to ensure that the outer edges of the lantern model did not overlap. This was rotated 2π radians around the y-axis.

As 'r' varies, 'h' must adjust to maintain the same volume of 0.4 m³. Therefore, it was important to find a relation between 'h' and 'r.' This process is shown below.

$$V = \pi \int_{-0.75h}^{0.25h} x^2 \mathrm{d}y \ = \pi \int_{-0.75h}^{0.25h} \left(r^2y^4 - 2r^2y^2 + r^2 \right) \mathrm{d}y$$

$$\frac{0.4}{\pi} = r^2 \left[\frac{y^5}{5} - \frac{2y^3}{3} + y \right]_{-0.75h}^{0.25h}$$

$$\frac{0.4}{\pi} = r^2 \left[\frac{(0.25h)^5}{5} - \frac{2(0.25h)^3}{3} + \left(0.25h\right) - \frac{(-0.75h)^5}{5} + \frac{2(-0.75h)^3}{3} - \left(-0.75h\right) \right]$$

$$\frac{0.4}{\pi} = r^2 \left[\frac{0.23828125h^5}{5} + \frac{0.40625h^3}{3} + h \right]$$

Using Wolfram Alpha, this was simplified to:

$$r = 0.356825 \sqrt{\frac{1}{0.0476563h^5 + 0.135417h^3 + h}}$$

Three examples of the model are shown below using different values of 'h.' All models in this investigation were plotted using CalcPlot3D[8].

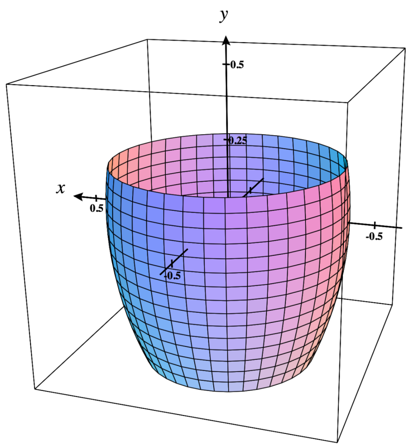

Figure 5 h=0.7, r=0.410807

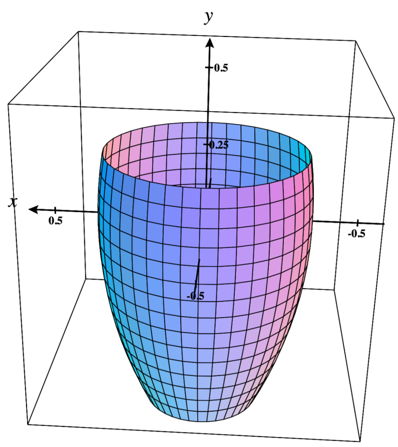

Figure 6 h=0.9, r=0.352127

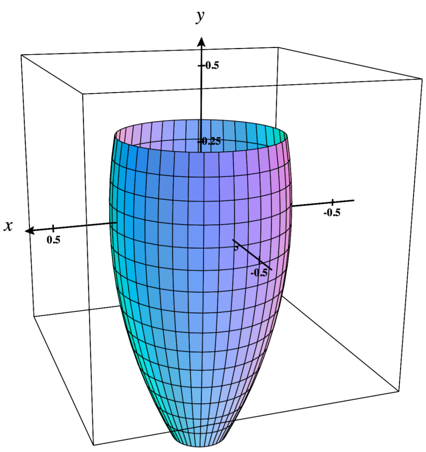

Figure 7 h=1.1, r=0.306314

[8] c3d.libretexts.org/CalcPlot3D/index.html
[11] sketch.io/sketchpad/en/

Following this, the next step was to minimise surface area. Surface area 'S' is given by[9]:

$$S = 2\pi \int x\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y + \pi\left(r(0.25h)^2 - r\right)^2$$

Here, $\pi\left(r(0.25h)^2 - r\right)^2$ is added to account for the fact that the top of the lantern is not open. Since the x-value at $y = 0.25h$ represents the radius ('R') at the top of the lantern, the area of the 'roof' is given by substituting this into the expression πR^2.

$$S = 2\pi \int_{-0.75h}^{0.25h} r\left(y^2 - 1\right)\sqrt{1 + 4r^2y^2}\,\mathrm{d}y + \pi r^2\left((0.25h)^4 - 2(0.25h)^2 + 1\right)$$

Since this was too difficult to simplify manually, it was entered into Wolfram Mathematica[10] with 'r' substituted in terms of 'h.' Following this, a graph plotting 'S' against 'h' was generated. However, as shown in figure 8, this did not yield any results due to the involvement of hypergeometric 2F1. The code I wrote for this may be found in the appendix.

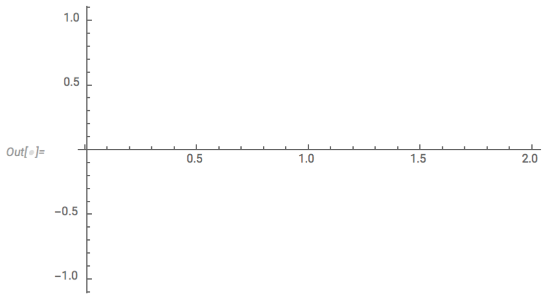

Figure 8 'S' against 'h'; h=0.01 to h=2

Euler's method also could not be used due to the fact that the integral contains variables, which cannot be easily manipulated using spreadsheets. Hence, this method was unfortunately found to be unsuccessful at the level of this investigation.

Model 2: Sphere on paraboloid

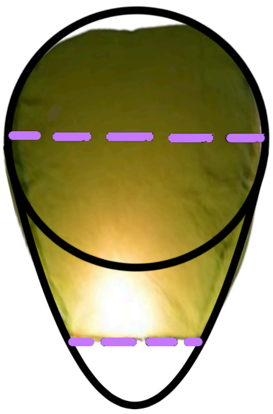

Figure 9 A sphere and paraboloid traced over a lantern

Another model that resembles the shape of a floating lantern can be created using a sphere and an elliptic paraboloid. This is more accurate than the solid of revolution as it models the dome-shaped top of the lantern more closely. Moreover, the sloped sides of the lantern towards the bottom also appear to be slightly more accurate as they are generally not as curved as the revolved quadratic model suggests.

In figure 9, the black lines show a cross-section of the intended model. While the sphere cross-section is drawn completely to indicate its shape, the domain of the sphere would actually be restricted to the the first purple line, which coincides with its diameter. Similarly, the domain of the paraboloid would be between the purple dotted lines.

This model was planned as shown in figure 10 on the next page.

[9] tutorial.math.lamar.edu/Classes/CalcII/SurfaceArea.aspx
[10] mathematica.wolframcloud.com/

The sphere in the model was defined using the following implicit equation[12]:

$$\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}=1\ ;\ 0\le z\le r$$

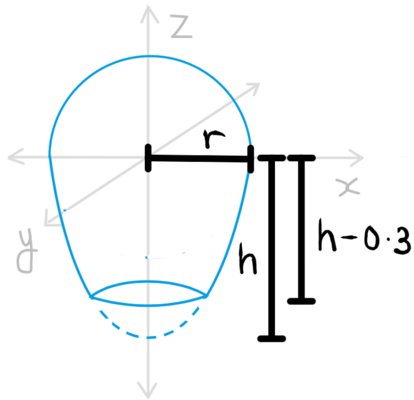

Figure 10 Defining parameters for sphere on paraboloid model

In order to derive the implicit equation of the paraboloid, the following process was used, where 'a' was to be found[13]:

$$z=\frac{x^2}{a}+\frac{y^2}{a}-h \qquad [at\ z=0,\ x^2+y^2=r^2]$$

$$0=\frac{x^2+y^2}{a}-h \qquad ha=x^2+y^2 \qquad ha=r^2 \qquad a=\frac{r^2}{h}$$

$$\boxed{z=h\left(\frac{x^2}{r^2}+\frac{y^2}{r^2}-1\right);-h+0.3\le z\le 0}$$

Similar to model 1, it was important to find a relation between 'r' and 'h' so that the lantern could maintain the same volume 'V', which is given by:

$$V=V_{hemisphere}+V_{paraboloid}=\frac{4}{6}\pi r^3+V_{paraboloid}$$

In order to find the volume of the paraboloid, the formula below was used[14] since the shape is rotationally symmetrical around the z-axis. Here, 'h' is the total length of the paraboloid and 'R' is the largest radius, which is found at the top-most circular cross-section.

$$V_{paraboloid}=\frac{1}{2}\pi R^2 h$$

As this paraboloid is only defined between z=-h+0.3 and z=0, the remaining part of the paraboloid underneath z=-h+0.3 had to be subtracted.

$$V_{paraboloid}=\frac{1}{2}\pi r^2 h-\frac{1}{2}\pi\left(\text{radius at } z=-h+0.3\right)^2\left(0.3\right)$$

The steps taken to find the radius at z=-h+0.3 are shown below, where the radius to be found is denoted by 'p.'

$$-h+0.3=h\left(\frac{2p^2}{r^2}-1\right)$$

$$\frac{-h+0.3}{h}+1=\frac{2p^2}{r^2}$$

$$\frac{r^2}{2}\left(\frac{-h+0.3}{h}+1\right)=p^2$$

$$\boxed{p=\sqrt{\frac{r^2}{2}\left(\frac{-h+0.3}{h}+1\right)}}$$

Only the positive root is taken since radius cannot be negative.

Hence, the volume of the paraboloid can now be simplified using Wolfram Alpha as shown:

$$V_{paraboloid}=\frac{\pi r^2}{2}\left(h-\frac{0.3}{2}\left(\frac{-h+0.3}{h}+1\right)\right)$$

$$\boxed{V_{paraboloid}=\frac{\pi h r^2}{2}-\frac{0.0706858r^2}{h}}$$

[12] mathworld.wolfram.com/Ellipsoid.html
[13] mathworld.wolfram.com/EllipticParaboloid.html
[14] www.mathcurve.com/surfaces.gb/paraboloidrevolution/paraboloidrevolution.shtml

Following this, the total volume 'V' of the lantern can be given by:

$$V = \frac{2}{3}\pi r^3 + \frac{h}{2}\pi r^2 - \frac{0.0706858}{h} r^2$$

Using Wolfram Alpha, the relation between 'r' and 'h' was simplified to be:

$$h = \frac{1}{r^2}\Big[-0.666667r^3 + \left(2.68786 \times 10^{-14}\right) \times \sqrt{\left(6.15181 \times 10^{26}\right)r^6 + \left(6.2287 \times 10^{25}\right)r^4 - \left(2.3498 \times 10^{26}\right)r^3 + \left(2.2439 \times 10^{25}\right)} + 0.127324\Big]$$

Three examples of the sphere on paraboloid model are shown on below using different values of 'r,' which produced different values of 'h.'

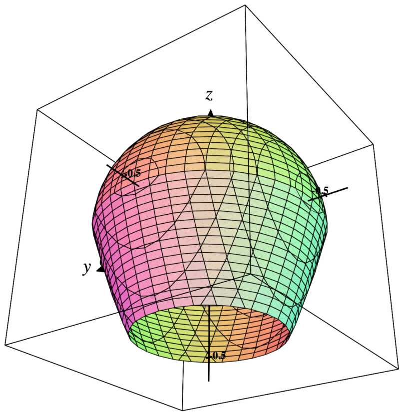

Figure 11 r=0.45, h=0.7200

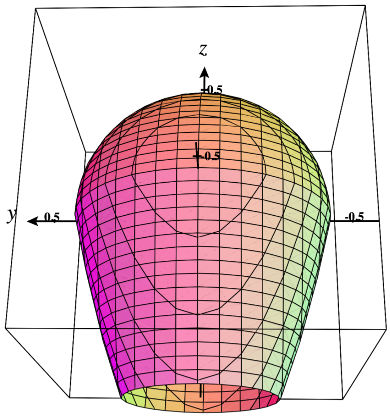

Figure 12 r=0.425, h=0.8935

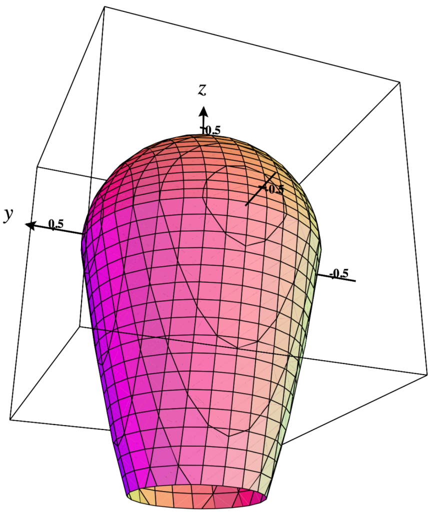

Figure 13 r=0.4, h=1.09916

The next step was to find the surface area of the lantern. Since the surface area of the hemisphere is known to be $2\pi r^2$, the total surface area 'S' can be represented using the equation:

$$S = 2\pi r^2 + S_{paraboloid}$$

The process used to find the surface area of the paraboloid is shown below[15].

$$for\ z = f(x, y),\ S_{paraboloid} = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy \qquad \left[z = \frac{hx^2}{r^2} + \frac{hy^2}{r^2} - h\right]$$

$$S_{paraboloid} = \int_{-r}^{r}\int_{-r}^{r} \sqrt{1 + \left(\frac{2hx}{r^2}\right)^2 + \left(\frac{2hy}{r^2}\right)^2}\, dx\, dy$$

Here, the equation includes the section below z=-h+0.3, which needed to be subtracted. The radius at z=-h+0.3 is denoted using 'p,' as it was before. Since this smaller section is geometrically similar to the larger paraboloid, 'r' and 'h' are directly substituted with 'p' and 0.3 respectively and placed into the same formula. The altered equation for 'S' which subtracts this section is shown on the next page.

[15] www.quora.com/How-would-you-calculate-the-area-of-a-paraboloid-z-x-2+y-2-with-1-le-z-le-4

$$S_{paraboloid} = \int_{-r}^{r}\int_{-r}^{r}\sqrt{1+\left(\frac{2hx}{r^2}\right)^2+\left(\frac{2hy}{r^2}\right)^2}\,dx\,dy - \int_{-p}^{p}\int_{-p}^{p}\sqrt{1+\left(\frac{2(0.3)x}{p^2}\right)^2+\left(\frac{2(0.3)y}{p^2}\right)^2}\,dx\,dy$$

In order to find the minimum surface area, values of 'r' were taken from 0.05 to 0.525, with intervals of 0.025. Values of 'r' beyond 0.525 were not taken because they produced values of 'h' which were less than 0.3. Since the domain of the paraboloid is between z=-h+0.3 and z=0, these values of 'h' produced errors in the model.

Using the relation found earlier, corresponding 'h' values were produced for each value of 'r.' This process was carried out using a spreadsheet, which can be found in the appendix. These were then used to find 'p', and all three variables (r, h, and p) were substituted into the equation above. The equation was then evaluated using Wolfram Alpha, and the resulting values of '$S_{paraboloid}$' were noted down. A part of this spreadsheet is shown below, the rest of which is found in the appendix.

r	h	S<paraboloid>
0.050	101.7927551	31.15307577000000000
0.075	45.1716541	20.73116230000000000
0.100	25.3331883	15.49386570000000000
0.125	16.1335598	12.32335490000000000
0.150	11.1217112	10.18087820000000000

Figure 14 Spreadsheet used to find 'S'

Following this, '$S_{paraboloid}$' was added to the surface area of the hemisphere, given by:

$$S_{hemisphere} = 2\pi r^2$$

This produced the total surface area 'S' of the lantern.

To find the value of 'r' which resulted in the minimum value of 'S,' the graph below was plotted.

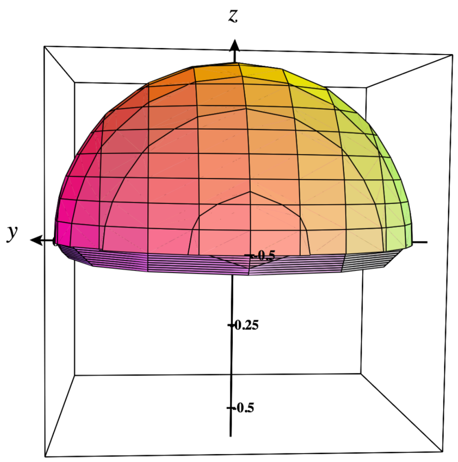

Figure 15 The optimal lantern

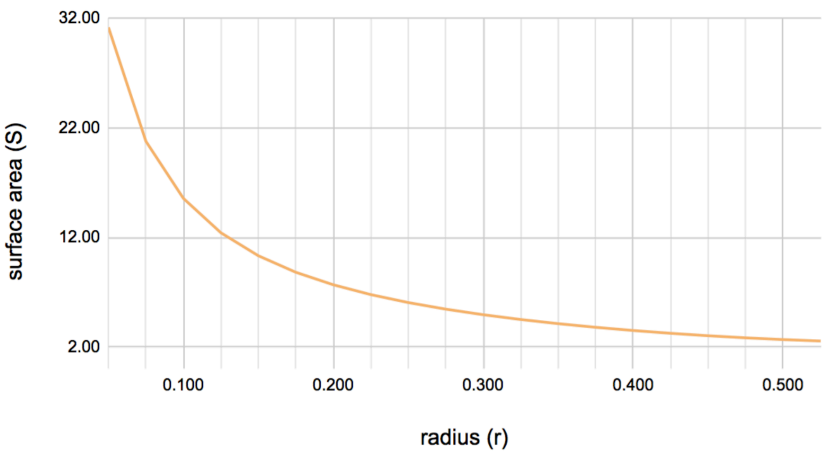

Figure 16 'S' plotted against 'r'

The graph shown in figure 16 suggests that the optimal surface area is found at the largest possible radius, where r=*0.525* and h=*0.3518073*. When plotted, these values model the lantern in figure 15, which takes on a rather unconventional shape. While it may be mathematically optimal, it is difficult to determine whether it would be physically efficient. In order to prevent the model from recognising such lanterns as optimal, variables may need to be restricted to different ranges, or physical formulae may need to be taken into account.

Model 3: Inverted teardrop

The inverted teardrop model is perhaps the most accurate, since it closely matches the slightly flattened top of the lantern, as well as the curve of the sides. In some ways, it also provides a more simplistic approach than the second model as it only requires one surface.

Figure 17 Tracing an inverted teardrop over the image of a lantern

On figure 17, the cross-sectional shape of a teardrop is traced over the image of a floating lantern. As seen on the figure, the black lines indicate the teardrop shape, and the purple line shows where the domain should be restricted. This figure brings out the accuracy with which the teardrop model fits onto the typical form of a lantern, as there are no large disparities visible between the edges of the lantern and the outline of the teardrop.

The model is then planned as shown on figure 18. Here, the height parameter 'h' is defined as the total length of the teardrop. However, the domain along the z-axis is restricted to be above -$0.25h$. The variable 'r' is taken to be the radius that coincides with the x-y plane.

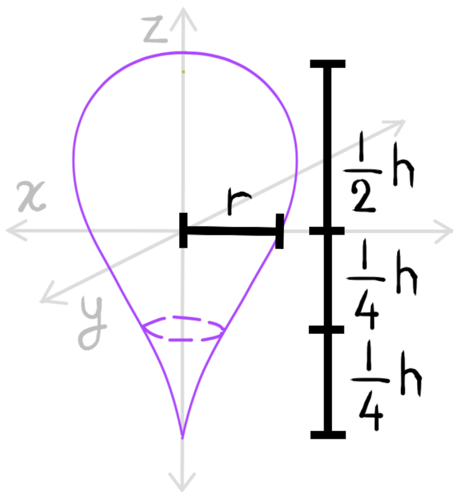

Figure 18 Defining parameters for the inverted teardrop model

In order to define the teardrop model, the following parametric equations were used[16]:

$$x = r(1-\cos(\theta))\sin(\theta)\cos(\varphi)$$
$$y = r(1-\cos(\theta))\sin(\theta)\sin(\varphi)$$
$$z = -\frac{h}{2}\cos(\theta)$$

[In the parametric for 'z', the coefficient is negative to invert the teardrop.]

In the equations, 'φ' had to be between 0 and π so that the teardrop rotated fully around the z-axis. However, the domain of 'θ' had to be determined as it would influence the domain along the z-axis, which needed to be restricted as shown in Figure 18. The steps taken to do this are shown below.

$$z = -\frac{h}{2}\cos\theta \qquad z \geq -\frac{h}{4} \qquad -\frac{h}{2}\cos\theta \geq -\frac{h}{4} \qquad \cos(\theta) \leq \frac{1}{2}$$

This results in $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$, which signify the end-points of the domain of θ. By verifying pairs of θ and 'z' values, the domains of θ and φ were determined to be:

$$0 \leq \varphi \leq \pi \quad \text{and} \quad \frac{\pi}{3} \leq \theta \leq \frac{5\pi}{3}$$

The next step was to find the volume so that a relation between 'r' and 'h' could be drawn. To simplify the process, the model was integrated using the disc method as shown next[17].

[16] paulbourke.net/geometry/teardrop/
[17] www.mathalino.com/reviewer/derivation-formulas/derivation-formula-volume-sphere-integration

Since all horizontal cross-sections are circular in shape, they can be treated as discs which make up the total volume 'V' of the model. The radius of each disc would be its corresponding 'x' value and the height of each disc would be a small difference in 'z.' Hence,

$$\mathrm{d}V = \pi x^2 \mathrm{d}z$$

$$V = \pi \int_{-0.25h}^{0} x^2 \mathrm{d}z + \pi \int_{0}^{0.5h} x^2 \mathrm{d}z$$

Here, the integral was split to avoid errors. To evaluate this, 'x' would have to be found in terms of 'z.' This was carried out as shown below.

$$x = r(1 - \cos\theta)\sin\theta$$

$$z = -\frac{h}{2}\cos\theta \qquad \text{where} \qquad 0 \le \theta \le 2\pi$$

Using the parametric for 'z':

$$-\frac{2z}{h} = \cos\theta \qquad \theta = arc\cos\left(-\frac{2z}{h}\right)$$

$$\boxed{x = r\left(1 + \frac{2z}{h}\right)\sin\left(\arccos\left(-\frac{2z}{h}\right)\right)}$$

This can now be substituted into the integral shown above.

$$V = \pi \int_{-0.25h}^{0} \left[r\left(1 + \frac{2z}{h}\right)\sin\left(\arccos\left(-\frac{2z}{h}\right)\right)\right]^2 \mathrm{d}z + \pi \int_{0}^{0.5h} \left[r\left(1 + \frac{2z}{h}\right)\sin\left(\arccos\left(-\frac{2z}{h}\right)\right)\right]^2 \mathrm{d}z$$

With Wolfram Alpha, this was simplified to $V = 2.474hr^2$. This is useful since it can be used to approximate the volume of similar teardrop shapes with the same parameters. Hence, using the equation for 'V' and the known value of 0.4 m^3, the relation between 'r' and 'h' was found to be:

$$h = \frac{V}{2.474r^2} = \frac{0.4}{2.474r^2}$$

$$\boxed{h = \frac{0.16168148747}{r^2}}$$

Using this relation, three examples of the teardrop model are shown below.

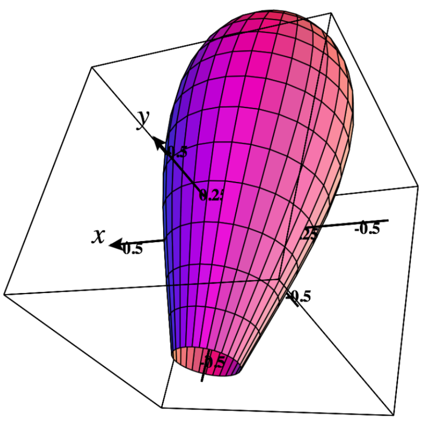

Figure 19 r=0.28, h=2.06226

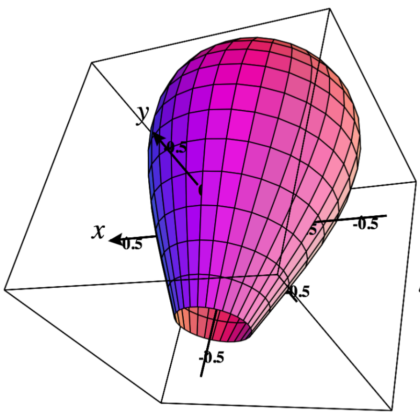

Figure 20 r=0.32, h=1.57892

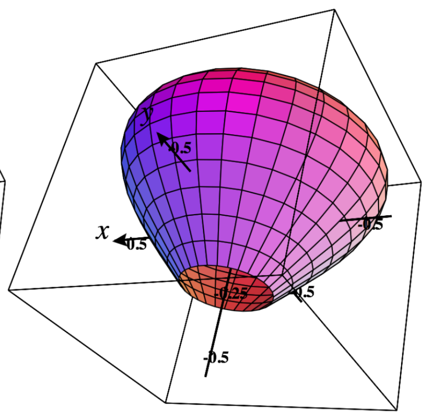

Figure 21 r=0.4, h=1.01051

Finally, the surface area 'S' of the model had to be found and optimised. The process carried out to find 'S' in terms of 'r' and 'h' is shown on the next page[18].

[18] www.youtube.com/watch?v=IdVILLByihs&t=22s

Since the model is a parameterised surface, it can be defined in the following notation:

$\vec{r} = < f(\theta, \varphi), g(\theta, \varphi), h(\theta, \varphi) >$

where f, g, and h are the parametric functions of 'x', 'y', and 'z' respectively.

Please note that the vector $\vec{r}$ is not the same as the radius parameter 'r.' Here, $\vec{r}$ represents a general position vector and can be distinguished from 'r' by the arrow above it.

Hence, the surface area of the model can be found using the formula:

$$S = \iint_R ||\vec{r}_\theta \times \vec{r}_\varphi|| \, d\theta \, d\varphi \quad \text{where} \quad \vec{r}_\theta = \left\langle \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right\rangle \quad , \quad \vec{r}_\varphi = \left\langle \frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi} \right\rangle$$

The partial derivatives taken to produce $\vec{r}_\theta$ and $\vec{r}_\varphi$ are shown below.

$$\frac{\partial x}{\partial \theta} = r\cos\varphi\cos\theta - r\cos\varphi \qquad \frac{\partial x}{\partial \varphi} = -r\left(1 - \cos\theta\right)\sin\theta\sin\varphi$$

$$\frac{\partial y}{\partial \theta} = r\sin\varphi\cos\theta - r\sin\varphi \qquad \frac{\partial y}{\partial \varphi} = r\left(1 - \cos\theta\right)\sin\theta\cos\varphi$$

$$\frac{\partial z}{\partial \theta} = \frac{h}{2}\sin\theta \qquad \frac{\partial z}{\partial \varphi} = 0$$

$$\vec{r}_\theta = \begin{pmatrix} rcos\varphi cos\theta - rcos\varphi \\ rsin\varphi cos\theta - rsin\varphi \\ (0.5h)\, sin\theta \end{pmatrix} \qquad \vec{r}_\varphi = \begin{pmatrix} -r(1 - cos\theta)\, sin\theta sin\varphi \\ r(1 - cos\theta)\, sin\theta cos\varphi \\ 0 \end{pmatrix}$$

The next step was to find the cross product of $\vec{r}_\theta$ and $\vec{r}_\varphi$ and its magnitude, which would become the integrand in the equation for 'S.' This was simplified using Wolfram Alpha.

$$\vec{r}_\theta \times \vec{r}_\varphi = \begin{pmatrix} r\cos\varphi\cos\theta - r\cos\varphi \\ r\sin\varphi\cos\theta - r\sin\varphi \\ (0.5h)\sin\theta \end{pmatrix} \times \begin{pmatrix} -r(1-\cos\theta)\sin\theta\sin\varphi \\ r(1-\cos\theta)\sin\theta\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -(0.5hr)\sin^2\theta(1-\cos\theta)\cos\varphi \\ -(0.5hr)\sin^2\theta(1-\cos\theta)\sin\varphi \\ -r^2\sin\theta(\cos\theta - 1)^2 \end{pmatrix}$$

$$\left\|\vec{r}_\theta \times \vec{r}_\varphi\right\| = \sqrt{[-(0.5hr)\sin^2\theta(1-\cos\theta)\cos\varphi]^2 + [-(0.5hr)\sin^2\theta(1-\cos\theta)\sin\varphi]^2 + [-r^2\sin\theta(\cos\theta - 1)^2]^2}$$

$$||\vec{r}_\theta \times \vec{r}_\varphi|| = 0.5r\sqrt{\sin^2\theta(\cos\theta - 1)^2\left[h^2\sin^2\theta + 4r^2(\cos\theta - 1)^2\right]}$$

This can now be placed in the equation for 'S' as shown below.

$$S = 0.5r\int_0^\pi \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \sqrt{\sin^2\theta(\cos\theta - 1)^2\left[h^2\sin^2\theta + 4r^2(\cos\theta - 1)^2\right]}\, d\theta \, d\varphi$$

Here, the limits have also been defined — the limits of the inner integral cover the range of values of θ, and the limits of the outer integral cover the range of values of φ used in the model.

In order to evaluate the double integral, a spreadsheet was used, similar to model 2. Values of 'r' were taken from 0.05 to 1, with intervals of 0.05. Then, 'h' was calculated for each value of 'r' using the relation found previously. Each pair of 'r' and 'h' values was substituted into the integral and entered into Wolfram Alpha. A part of the spreadsheet used (found in the appendix) is shown in figure 22.

r	h	inner integral	S
0.05	64.67259499	4.78648	15.0371704
0.1	16.16814875	2.39418	7.521538299
0.15	7.185843888	1.60245	5.034245148
0.2	4.042037187	1.22392	3.845058081
0.25	2.5869038	1.03202	3.24218645

Figure 22 Spreadsheet used to evaluate 'S' for model 3

Following this, a graph plotting 'S' against 'r' was generated, which would visually indicate the 'r' value which produces the minimum value of 'S.' This graph is shown below in figure 24.

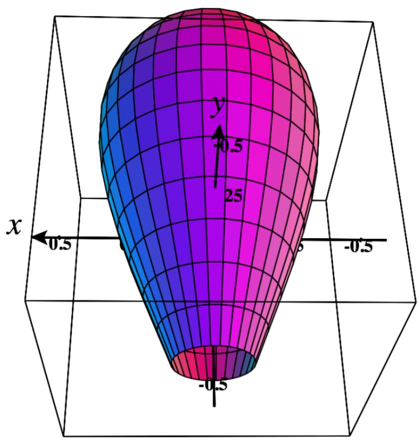

Figure 23 r=0.3, h=1.796460972

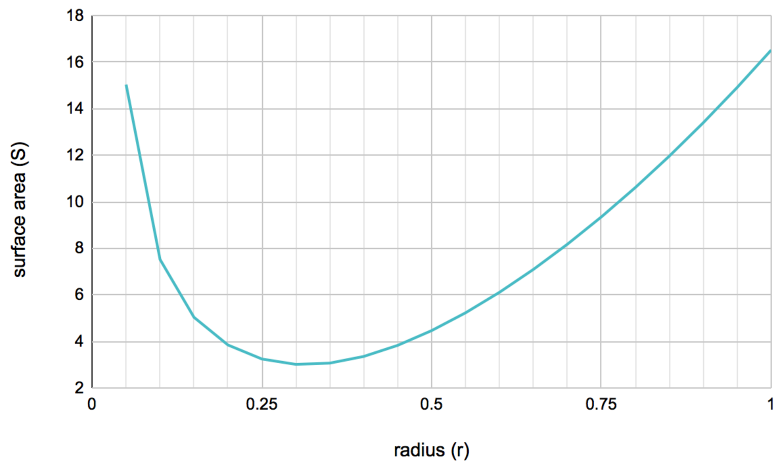

Figure 24 Plotting 'S' against 'r'

Figure 23 shows the optimal configuration as determined through this model, where the surface area of the lantern is minimal. Hence, this method was successful in optimising floating lanterns.

Evaluation

Two out of three models used in this investigation successfully yielded results. The two optimal values of 'r' found, which were 0.525 metres and 0.3 metres, had a significant difference between them. However, this is understandable since the lantern modelled by r=0.525m did not resemble any realistic shape of a floating lantern. However, the accuracy of these results cannot be verified since producing and testing real floating lanterns would be difficult and wasteful.

Since spreadsheets were used for models 2 and 3, only a small number of 'r' values could be taken into account. Hence, the results are only approximations that indicate the interval of 'r' closest to the optimal value. This may have been overcome by the use of more sophisticated computation tools or techniques.

While models 2 and 3 resemble the shape of a lantern as closely as possible at the level of this investigation, there are still nuances that were not accurately reflected in the models. For example, most lanterns have a circular cross-section at the bottom and a square cross-section at the top, but this could not be taken into account.

Conclusion

This investigation provides an insight into the approximate shape of an optimal floating lantern. It also indicates which mathematical models can be used most effectively for this problem. Surprisingly, model 1 failed to produce an answer due to the emergence of a hypergeometric function, while model 3 brought forth an elegant solution. This shows that fundamental and simple models do not necessarily provide the simplest answers. In this case, the best choice was the teardrop model, which not only simulated the shape of a lantern most accurately, but also brought the practical benefits of a simple and streamlined calculation process.

The models used in the investigation can also be applied to other similar situations. Perhaps the model that could lend itself to the largest range of problems would be the solid of revolution since it can be used to simulate any shape with circular cross-sections. Quadratic solids of revolution in particular could be used to model objects such as Chinese lanterns and vases. Similarly, model 3 would be extremely useful in modelling hot-air balloons, which are almost exactly the same shape as an inverted teardrop.

Overall, floating lanterns were an interesting topic to explore as they allowed me to delve into a range of new mathematical concepts such as surface integrals and parametric equations. This also allowed me to engage with the link between maths and real-world objects, which was a valuable and rewarding process.

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Appendix

Code used to simplify 'S' for model 1:

```
Integrate[2pi*(0.356825*Sqrt[1/
(0.0476563(h^5)+0.135417(h^3))])*(y^2-1)*Sqrt[1+4((0.356825*Sqrt[1/
(0.0476563(h^5)+0.135417(h^3))])^2)(y^2)],{y,-0.75h,0.25h}]+(pi*(0.356825*Sqrt[1/
(0.0476563(h^5)+0.135417(h^3))])^2*((0.25h)^4-2*(0.25h)^2+1))
```

Code used to plot 'S' against 'h' for model 1:

```
Plot[(0.127324080625` (1-0.125` h^2+0.00390625` h^4) pi)/(0.135417` h^3+0.0476563`
h^5)+0.71365 Sqrt[1/(0.135417` h^3+0.0476563` h^5)] pi (0.0175449 h (h^2 (6.01136 +2.84153
h+1. h^3) Sqrt[1.` +6.011360122507414`/(2.8415340678986833` h+1.` h^3)]+(-42.7474-2.84153
h^3-1. h^5) Hypergeometric2F1[-0.5,0.5,1.5,-(6.011360122507414`/(2.8415340678986833` h+1.`
h^3))])+0.0058483 h (h^2 (0.667929 +2.84153 h+1. h^3) Sqrt[1.` +0.6679289025008237`/
(2.8415340678986833` h+1.` h^3)]+(-42.7474-2.84153 h^3-1. h^5)
Hypergeometric2F1[-0.5,0.5,1.5,-(0.6679289025008237`/(2.8415340678986833` h+1.` h^3))])),{h,
0.01,2}]
```

Spreadsheet used to calculate values of 'S' for model 2:

r	h	p	S<r>	S<p>	S<paraboloid>	S<hemisphere>	S<total>
0.050	101.7927551	0.001919	31.15660000000000000	0.00352423000000000	31.15307577000000000	0.02	31.17
0.075	45.1716541	0.004322	20.73910000000000000	0.00793770000000000	20.73116230000000000	0.04	20.77
0.100	25.3331883	0.007695	15.50800000000000000	0.01413430000000000	15.49386570000000000	0.06	15.56
0.125	16.1335598	0.012053	12.34550000000000000	0.02214510000000000	12.32335490000000000	0.10	12.42
0.150	11.1217112	0.017420	10.21290000000000000	0.03202180000000000	10.18087820000000000	0.14	10.32
0.175	8.0872505	0.023833	8.66483000000000000	0.04384630000000000	8.62098370000000000	0.19	8.81
0.200	6.1068892	0.031345	7.47904000000000000	0.05773960000000000	7.42130040000000000	0.25	7.67
0.225	4.7395686	0.040028	6.53224000000000000	0.07387250000000000	6.45836750000000000	0.32	6.78
0.250	3.7530174	0.049980	5.75078000000000000	0.09248510000000000	5.65829490000000000	0.39	6.05
0.275	3.0154966	0.061334	5.08821000000000000	0.11391400000000000	4.97429600000000000	0.48	5.45
0.300	2.4478014	0.074264	4.51406000000000000	0.13862100000000000	4.37543900000000000	0.57	4.94
0.325	2.0000314	0.089004	4.00792000000000000	0.16724900000000000	3.84067100000000000	0.66	4.50
0.350	1.6395367	0.105865	3.55607000000000000	0.20070200000000000	3.35536800000000000	0.77	4.13
0.375	1.3443030	0.125265	3.14965000000000000	0.24026300000000000	2.90938700000000000	0.88	3.79
0.400	1.0991561	0.147766	2.78365000000000000	0.28778100000000000	2.49586900000000000	1.01	3.50
0.425	0.8935120	0.174134	2.45637000000000000	0.34598600000000000	2.11038400000000000	1.13	3.25
0.450	0.7200197	0.205393	2.16943000000000000	0.41893500000000000	1.75049500000000000	1.27	3.02
0.475	0.5737351	0.242875	1.92764000000000000	0.51267700000000000	1.41496300000000000	1.42	2.83
0.500	0.4515781	0.288170	1.73848000000000000	0.63601700000000000	1.10246300000000000	1.57	2.67
0.525	0.3518073	0.342809	1.60991000000000000	0.80078800000000000	0.80912200000000000	1.73	2.54

Spreadsheet used to calculate values of 'S' for model 3:

r	h	inner integral	S
0.05	64.67259499	4.78648	15.0371704
0.1	16.16814875	2.39418	7.521538299
0.15	7.185843888	1.60245	5.034245148
0.2	4.042037187	1.22392	3.845058081
0.25	2.5869038	1.03202	3.24218645
0.3	1.796460972	0.958637	3.011646957
0.35	1.319848877	0.975946	3.066024784
0.4	1.010509297	1.06707	3.352299273
0.45	0.7984270986	1.21879	3.82894171
0.5	0.6467259499	1.42053	4.462726612
0.55	0.5344842561	1.6644	5.228866813
0.6	0.449115243	1.94493	6.1101778
0.65	0.3826780769	2.25847	7.09519276
0.7	0.3299622193	2.60271	8.176654615
0.75	0.2874337555	2.97615	9.349850976
0.8	0.2526273242	3.37784	10.61179733
0.85	0.2237806055	3.80715	11.96051447
0.9	0.1996067747	4.26367	13.39471435
0.95	0.1791484626	4.74713	14.91354873
1	0.1616814875	5.25734	16.51642072